逆 z 変換

Inverse z transform

逆 z 変換

$$x(k) = Z^{-1}[X(z)] = \frac{1}{2\pi j}\oint X(z)z^{k-1}\mathrm{d}z$$

$\oint$ は原点を中心とし，半径rの円を反時計回りに回る周回積分を意味する.

10.2 部分分数展開による逆 z 変換

$$X(z) = \frac{3 - 1.25z^{-1}}{(1 - 0.5z^{-1})(1 - 0.25z^{-1})}$$

$$X(z) = \frac{3z^2 - 1.25z}{(z - 0.5)(z - 0.25)} = \frac{z(3z - 1.25)}{(z - 0.5)(z - 0.25)}$$

$$\frac{X(z)}{z} = \frac{3z - 1.25}{(z - 0.5)(z - 0.25)} = \frac{\alpha}{z - 0.5} + \frac{\beta}{z - 0.25}$$

$$\alpha = (z - 0.5)\frac{X(z)}{z}\bigg|_{z=0.5} = 1$$

$$\beta = (z - 0.25)\frac{X(z)}{z}\bigg|_{z=0.25} = 2$$

$$X(z) = \frac{z}{z - 0.5} + \frac{2z}{z - 0.25} = \frac{1}{1 - 0.5z^{-1}} + \frac{2}{1 - 0.25z^{-1}}$$

$$x(k) = Z^{-1}[X(z)] = (0.5^k + 2 \cdot 0.25^k)u_s(k)$$

10.2 部分分数展開による逆 z 変換

$$X(z) = \frac{z^{-2}}{(1-0.5z^{-1})(1-z^{-1})^2}$$

$$X(z) = \frac{z}{(z-0.5)(z-1)^2}$$

$$\frac{X(z)}{z} = \frac{1}{(z-0.5)(z-1)^2} = \frac{\alpha}{z-0.5} + \frac{\beta_1}{(z-1)^2} + \frac{\beta_2}{z-1}$$

$$\alpha = 4$$

$$\beta_1 = (z-1)^2 \frac{X(z)}{z}\bigg|_{z=1} = 2$$

$$\beta_2 = \frac{d}{dz}(z-1)^2 \frac{X(z)}{z}\bigg|_{z=1} = \frac{d}{dz}\frac{1}{z-0.5}\bigg|_{z=1} = -4$$

$$X(z) = \frac{4z}{z-0.5} + \frac{2z}{(z-1)^2} - \frac{4z}{z-1}$$

$$x(k) = Z^{-1}[X(z)] = (4 \cdot 0.5^k + 2k - 4)u_s(k)$$

10.3 逆 z 変換の性質

線形性

二つの離散時間信号 $x(k)$, $y(k)$の z 変換を
それぞれ $X(z), Y(z)$ とするとき,

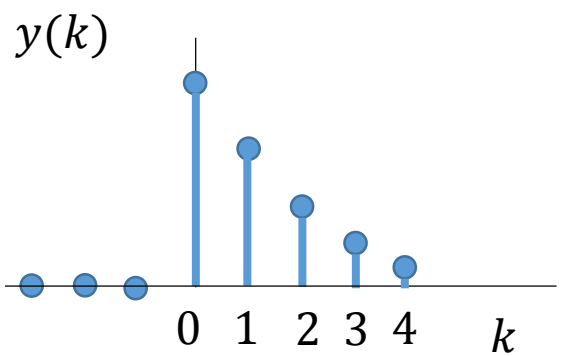
$$Z[ax(k) + by(k)] = aX(z) + bY(z)$$

が成り立つ．ただし，a, b は定数である.

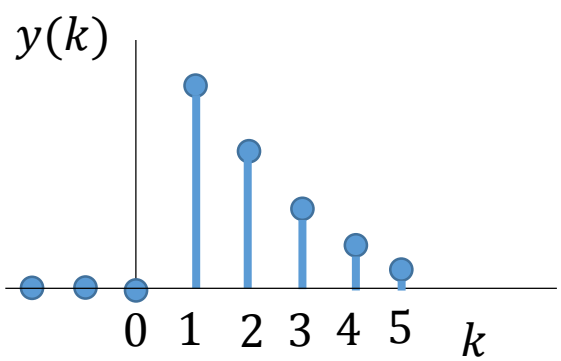
10.3 逆 z 変換の性質

時間軸推移（時間遅れ）

$$Z[x(k-m)] = z^{-m}X(z)$$



$X(z) = Z[x(k)]$



$Y(z) = Z[x(k-1)] = z^{-1}X(z)$

$$Z[x(k-1)] = z^{-1}X(z)$$

10.3 逆 z 変換の性質

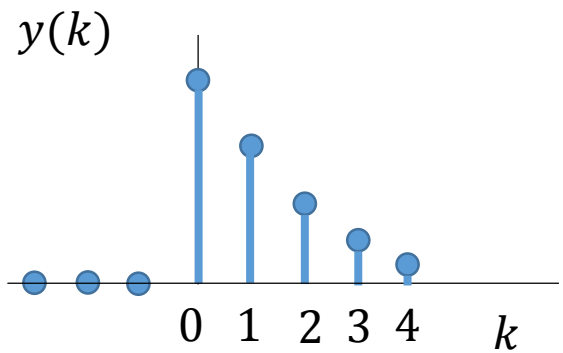
時間軸推移（時間進み）

$$Z[x(k+m)] = z^m X(z) - z^m x(0) - z^{m-1} x(1) - \cdots - zx(m-1)$$

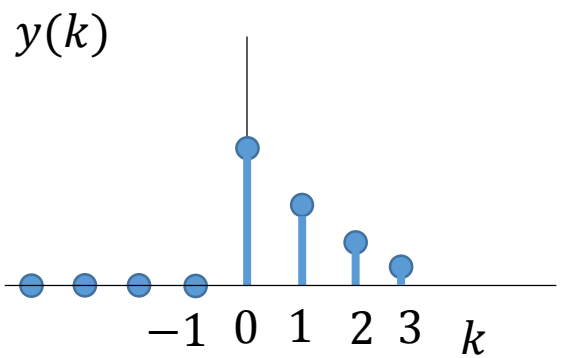
$$y(k) = x(k+1)u_s(k)$$

$$Y(z) = Z[x(k+1)] = \sum_{k=0}^{\infty} x(k+1) z^{-k} = z\left(x(0) + \sum_{k=0}^{\infty} x(k+1) z^{-(k+1)} - x(0)\right)$$

$$= zX(z) - zx(0)$$



$$X(z) = Z[x(k)]$$



$$Y(z) = Z[x(k+1)]$$
$$= zX(z) - zx(0)$$

10.3　逆 z 変換の性質

周波数軸推移

$$Z\left[e^{j\omega_0 k}x(k)\right] = X\left(e^{-j\omega_0}z\right)$$

時間軸反転

$$Z[x(-k)] = X\left(\frac{1}{z}\right)$$

z 領域における微分

$$Z[kx(k)] = -z\frac{\mathrm{d}X(z)}{\mathrm{d}z}$$

10.3　逆 z 変換の性質

離散時間信号のたたみ込み

$$Z[x(k) * y(k)] = X(z)Y(z)$$

$*$ は次式のたたみ込み和を表す.

$$x(k) * y(k) = \sum_{l=-\infty}^{\infty} x(l)y(k-l)$$

10.3　逆 z 変換の性質

初期値の定理

$$x(0) = \lim_{z \to \infty} X(z)$$

最終値の定理

$$x(\infty) = \lim_{k \to \infty} x(k) = \lim_{z \to 1} (1 - z^{-1}) X(z)$$

10.3　逆 z 変換の性質

和の z 変換

$$y(k) = \sum_{i=0}^{k} x(i)$$

$$Y(z) = \frac{1}{1 - z^{-1}} X(z)$$

差の z 変換

$$y(k) = x(k) - x(k-1)$$

$$Y(z) = X(z) - z^{-1}X(z) = (1 - z^{-1})X(z)$$

10.4 z 変換を用いた差分方程式の解法

$$x(k+2) - 0.8x(k+1) + 0.15x(k) = 1, \qquad k \geq 0$$

$$x(0) = x(1) = 1$$

$$(z^2X(z) - z^2 - z) - 0.8(zX(z) - z) + 0.15X(z) = \frac{z}{z-1}$$

$$(z^2 - 0.8z + 0.15)X(z) = \frac{z(z^2 - 0.8z + 0.8)}{z-1}$$

$$\frac{X(z)}{z} = \frac{z^2 - 0.8z + 0.8}{(z-1)(z-0.5)(z-0.3)}$$

$$\frac{X(z)}{z} = \frac{\alpha}{z-1} + \frac{\beta}{z-0.5} + \frac{\gamma}{z-0.3}$$

$$\alpha = \frac{20}{7} \qquad \beta = -\frac{13}{2} \qquad \gamma = \frac{65}{14}$$

$$X(z) = \frac{20}{7}\frac{1}{z-1} - \frac{13}{2}\frac{1}{z-0.5} + \frac{65}{14}\frac{1}{z-0.3}$$

$$x(k) = \frac{20}{7} - \frac{13}{2}0.5^k + \frac{65}{14}0.3^k$$